

# Latent solitons, black strings, black branes, and equations of state in Kaluza-Klein models

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In Kaluza-Klein models with an arbitrary number of toroidal internal spaces, we investigate soliton solutions which describe the gravitational field of a massive compact object. Each  $d_i$ -dimensional torus has its own scale factor  $C_i$ ,  $i = 1, \dots, N$ , which is characterized by a parameter  $\gamma_i$ . We single out the physically interesting solution corresponding to a point-like mass. For the general solution we obtain equations of state in the external and internal spaces. These equations demonstrate that the point-like mass soliton has dust-like equations of state in all spaces. We also obtain the parameterized post-Newtonian parameters, which give the possibility to obtain the formulas for perihelion shift, deflection of light and time delay of radar echoes. Additionally, the gravitational experiments lead to a strong restriction on the parameters of the model:  $\tau = \sum_{i=1}^N d_i \gamma_i = -(2.1 \pm 2.3) \times 10^{-5}$ . The point-like mass solution with  $\gamma_1 = \dots = \gamma_N = (1 + \sum_{i=1}^N d_i)^{-1}$  contradicts this restriction. The condition  $\tau = 0$  satisfies the experimental limitation and defines a new class of solutions which are indistinguishable from general relativity. We call such solutions latent solitons. Black strings and black branes with  $\gamma_i = 0$  belong to this class. Moreover, the condition of stability of the internal spaces singles out black strings/branes from the latent solitons and leads uniquely to the black string/brane equations of state  $p_i = -\varepsilon/2$ ,  $i = 1, \dots, N$ , in the internal spaces and to the number of the external dimensions  $d_0 = 3$ . The investigation of multidimensional static spherically symmetric perfect fluid with dust-like equation of state in the external space confirms the above results.

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## I. INTRODUCTION

Modern observational phenomena, such as dark energy and dark matter, are the great challenge for present cosmology, astrophysics and theoretical physics. Within the scope of the standard models, it has still not been offered a satisfactory explanation to these problems. This forces the search of solutions to these problems beyond the standard models, for example, by considering models with extra dimensions. This generalization follows from the modern theories of unification such as superstrings, supergravity, and M-theory, which have the most self-consistent formulation in spacetimes with extra dimensions. Obviously, these physical theories should be consistent with observations. In a previous paper [1], two of the present authors have examined the family of 5-dimensional soliton solutions (see [2–4]) which describes the gravitational field of massive compact objects in spacetimes with compact toroidal extra dimensions. Among these solutions, the one which corresponds to a point-like massive source has been singled out. At a first glance, this is a good physical approximation for astrophysical objects in the weak-field limit because it

works very well in general relativity. However, it has been found that such approach contradicts famous gravitational experiments (perihelion shift, light deflection, and time delay of radar echoes) in 5-dimensional spacetime [1, 5]. The only compact astrophysical objects that satisfy the observational data with the same accuracy as general relativity, are objects with a dust-like equation of state ( $p_0 = 0$ ) in our three dimensions and the equation of state  $p_1 = -\varepsilon/2$  in the fifth dimension. Black strings have such equations of state. Additionally, it was shown that these equations of state satisfy the necessary condition of the internal space stabilization (see also [6]). This is a very strong bound on the equations of state for the perfect fluid in 5-dimensional Kaluza-Klein models. Therefore, it is important to understand how common is this restriction. This was the main reason for our further investigations on the subject.

Multidimensional soliton solutions were found in Refs. [7–10]. In the present paper, we investigate the solution from Ref. [9] because, to our knowledge, it is the most general form of solitons in the case of an arbitrary number  $N$  of toroidal internal spaces. Our investigation shows that it is a very fruitful idea to consider the most general case. In particular, we obtain the most general form of the equations of state for a soliton matter source and we find an experimental restriction for the parameters of the general solution. However, the most important advantage of such approach is the discovery of a new class of solutions. These solutions satisfy the well known

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gravitational experiments at the same level of accuracy as general relativity. With these experiments only, it is impossible to differ these new solutions from general relativity. For this reason we call them *latent solitons*. Black strings and black branes belong to this class of solutions. We show that only black strings and black branes have equations of state in the internal spaces which do not spoil the condition of the internal space stabilization. All these conclusions are confirmed by a general analysis of the multidimensional static spherically symmetric perfect fluid with dust-like equation of state in the external (our) space. In the case of three-dimensional external space, such perfect fluid describes observable astrophysical objects (e.g., the Sun) in Kaluza-Klein models. Therefore, our investigation demonstrates that the condition of stability of the internal spaces (i) singles out black strings/branes among the latent solitons, (ii) leads uniquely to the black string/brane equations of state  $p_i = -\varepsilon/2$ ,  $i = 1, \dots, N$ , in the internal spaces, and (iii) to the number of the external dimensions  $d_0 = 3$ .

The remainder of this paper is structured as follows. In Sec. II we consider a family of the most general soliton solutions and single out one of these solutions which corresponds to the physically important case of a point-like massive source. In Sec. III we obtain equations of state for soliton matter sources in the most general case and show that the point-like massive solution has dust-like equations of state in the external space as well as in all internal spaces. In Sec. IV we find the parameterized post-Newtonian (PPN) parameters in the general case. These parameters give us the possibility to derive the formulas for perihelion shift, deflection of light and time delay of radar echoes. Using the constraint which comes from the Cassini spacecraft experiment, we obtain in the general case a strong restriction on the solitonic parameter  $\tau = -(2.1 \pm 2.3) \times 10^{-5}$ . The particular case  $\tau = 0$  defines a new class of latent solitons. We present the equations of state for latent solitons as well as for black branes, which are a particular case of latent solitons. In Sec. V, we investigate a multidimensional static spherically symmetric perfect fluid with dust-like equation of state ( $p_0 = 0$ ) in the external space and arbitrary equations of state ( $p_i = \omega_i \varepsilon$ ) in the internal spaces. We demonstrate that the agreement with gravitational experiments leads to the latent soliton equations of state in the internal spaces. The internal space stability condition singles out the black brane equations of state. Our main results are summarized in Sec. VI.

## II. SOLITON METRICS

As we have already pointed out in the Introduction, the 5-dimensional soliton solutions [2–4] have been generalized to an arbitrary number of dimensions in Refs. [7–10]. To our knowledge, the most general form of these solutions was given in Ref. [9], and in isotropic (with re-

spect to our three-dimensional space) coordinates it reads

$$\begin{aligned} ds^2 = & A(r_3) c^2 dt^2 + B(r_3) (dr_3^2 + r_3^2 d\Omega_2^2) \\ & + \sum_{i=1}^N C_i(r_3) ds_i^2 = \left( \frac{ar_3 - 1}{ar_3 + 1} \right)^{2\theta} c^2 dt^2 \\ & - \left( 1 - \frac{1}{a^2 r_3^2} \right)^2 \left( \frac{ar_3 + 1}{ar_3 - 1} \right)^{2\theta(1-\tau)} (dr_3^2 + r_3^2 d\Omega_2^2) \\ & - \sum_{i=1}^N \left( \frac{ar_3 + 1}{ar_3 - 1} \right)^{2\theta\gamma_i} ds_i^2, \end{aligned} \quad (1)$$

where  $r_3$  is the length of the radius vector in three-dimensional space,  $ds_i^2 = \sum_{j=1}^{d_i} d\xi_{(i)j}^2$  is the line element of the  $d_i$ -dimensional torus, and the parameters  $\tau$ ,  $\theta$  and  $\gamma_i$  satisfy the condition [11, 12]

$$\theta^2 [(\tau - 1)^2 + \sigma + 1] = 2, \quad \tau \equiv \sum_{i=1}^N d_i \gamma_i, \quad \sigma \equiv \sum_{i=1}^N d_i \gamma_i^2. \quad (2)$$

In the weak-field limit  $1/(ar_3) \ll 1$ , the metric coefficients are given by

$$A(r_3) \approx 1 - \frac{4\theta}{ar_3} + \frac{16\theta^2}{a^2} \frac{1}{2r_3^2}, \quad (3)$$

$$B(r_3) \approx -1 - \frac{4\theta(1-\tau)}{ar_3}, \quad (4)$$

$$C_i(r_3) \approx -1 - \frac{4\theta\gamma_i}{ar_3}. \quad (5)$$

These expansions will help us to define important properties of the soliton solution (1), for example, observational restrictions on the parameters of solitons and equations of state for the matter source. These formulas are also useful to single out the case of a point-like mass  $m$  at rest as a matter source.

In the weak-field limit, the line element of a point-like mass  $m$  at rest in a  $(1 + D)$ -dimensional spacetime with toroidal extra dimensions is [5]

$$\begin{aligned} ds^2 \approx & \left( 1 - \frac{r_g}{r_3} + \frac{r_g^2}{2r_3^2} \right) c^2 dt^2 \\ & - \left( 1 + \frac{1}{D-2} \frac{r_g}{r_3} \right) (dr_3^2 + r_3^2 d\Omega_2^2) \\ & - \left( 1 + \frac{1}{D-2} \frac{r_g}{r_3} \right) \sum_{i=1}^N ds_i^2, \end{aligned} \quad (6)$$

where  $r_g = 2G_N m/c^2$ , with  $G_N$  being the Newtonian gravitational constant. The comparison of the metric coefficients (3)-(5) with the corresponding metric coefficients in Eq. (6) shows that for the point-like mass, we have:

(i) The equality

$$\frac{4\theta}{a} = r_g \quad (7)$$

holds. It follows that  $\text{sign } a = \text{sign } \theta$ . Because the solution (1) is invariant under the simultaneous change  $a \rightarrow -a, \theta \rightarrow -\theta$ , we can choose  $a, \theta > 0$ .

(ii) The parameters  $\gamma_i$  should take the same value for all internal spaces, namely:

$$\gamma_1 = \gamma_2 = \dots = \gamma_N = \frac{1}{1 + D'}, \quad (8)$$

where  $D' = \sum_{i=1}^N d_i = D - 3$  is the total number of extra dimensions.

(iii) The parameters  $\theta$  and  $a$  are given by

$$\theta = \sqrt{\frac{2(1 + D')}{2 + D'}}, \quad a = \frac{4}{r_g} \sqrt{\frac{2(1 + D')}{2 + D'}}, \quad (9)$$

where we also took into account the constraint (2) and the relation (7). Therefore, Eqs. (7)-(9) completely define the point-like mass soliton, i.e., the solution where delta-shaped  $T_{00}$  is the only non-zero component of the energy-momentum tensor. To demonstrate it, in the next section we derive equations of state for the general soliton solution (1).

### III. EQUATIONS OF STATE. GENERAL CASE

It is worth noting that the dependence of the metric coefficients only on  $r_3$  in Eq. (1) means that the matter source for such metrics is uniformly “smeared” over extra dimensions [14, 15]. It is clear that in this case the non-relativistic gravitational potential depends only on  $r_3$  and exactly coincides with the Newtonian one. Since the function  $A(r_3)$  is the metric coefficient  $g_{00}$ , we obtain  $4\theta/a = r_g = 2G_N m/c^2$ , and the expansions (3)-(5) become

$$A(r_3) \approx 1 - \frac{r_g}{r_3} + \frac{1}{2} \frac{r_g^2}{r_3^2}, \quad (10)$$

$$B(r_3) \approx -1 - (1 - \tau) \frac{r_g}{r_3}, \quad (11)$$

$$C_i(r_3) \approx -1 - \gamma_i \frac{r_g}{r_3}. \quad (12)$$

From these expressions, we can easily get the perturbations  $h_{00} = -r_g/r_3, h_{\alpha\alpha} = -(1 - \tau)r_g/r_3$  and  $h_{\mu_i\mu_i} = -\gamma_i r_g/r_3$ , of the order of  $1/c^2$  over the flat spacetime, that gives us the possibility to find the components of Ricci tensor up to the same order, namely:

$$R_{00} \approx \frac{1}{2} \Delta h_{00} = \frac{1}{2} k_N m \delta(\mathbf{r}_3) c^2 = \frac{1}{2} k_N \rho_3 c^2, \quad (13)$$

$$R_{\alpha\alpha} \approx \frac{1}{2} \Delta h_{\alpha\alpha} = \frac{1}{2} (1 - \tau) k_N \rho_3 c^2, \quad \alpha = 1, 2, 3, \quad (14)$$

$$R_{\mu_i\mu_i} \approx \frac{1}{2} \Delta h_{\mu_i\mu_i} = \frac{1}{2} \gamma_i k_N \rho_3 c^2, \quad (15)$$

with

$$\mu_i = 1 + \sum_{j=0}^{i-1} d_j, \dots, d_i + \sum_{j=0}^{i-1} d_j; \quad i = 1, \dots, N,$$

where  $d_0 = 3, k_N \equiv 8\pi G_N/c^4$  and  $\Delta = \delta^{ik} \partial^2 / \partial x^i \partial x^k$  is the  $D$ -dimensional Laplace operator (see [5] for details). We also introduced the non-relativistic three-dimensional mass density  $\rho_3 = m\delta(\mathbf{r}_3)$ , which is connected with the  $D$ -dimensional mass density  $\rho_D = \rho_3/V_{D'}$ . Here,  $V_{D'}$  is the total volume of the internal spaces. For example, if the  $i$ -th torus has periods  $a_{(i)j}$ , then  $V_{D'} = \prod_{i=1}^N \prod_{j=1}^{d_i} a_{(i)j}$ .

Now, we want to define the components of the energy-momentum tensor with the help of Einstein equation

$$R_{ik} = \frac{2S_D \tilde{G}_D}{c^4} \left( T_{ik} - \frac{1}{D-1} g_{ik} T \right), \quad (16)$$

where  $S_D = 2\pi^{D/2}/\Gamma(D/2)$  is the total solid angle (surface area of the  $(D-1)$ -dimensional sphere of unit radius) and  $\tilde{G}_D$  is the gravitational constant in the  $(D+1)$ -dimensional spacetime. Introducing the quantity  $k_D \equiv 2S_D \tilde{G}_D/c^4$  and keeping in mind that we are considering compact astrophysical objects at rest in our three-dimensional space (what results in  $T_{11} = T_{22} = T_{33} = 0$ ), we arrive at the following Einstein equations:

$$\frac{1}{2} k_N \rho_3 c^2 \approx k_D \left( T_{00} - \frac{1}{D-1} T g_{00} \right), \quad (17)$$

$$\frac{1}{2} (1 - \tau) k_N \rho_3 c^2 \approx k_D \left( -\frac{1}{D-1} T g_{\alpha\alpha} \right), \quad (18)$$

$$\frac{1}{2} \gamma_i k_N \rho_3 c^2 \approx k_D \left( T_{\mu_i\mu_i} - \frac{1}{D-1} T g_{\mu_i\mu_i} \right). \quad (19)$$

Therefore, the required components of the energy-momentum tensor are

$$T_{00} \approx \frac{k_N V_{D'}}{k_D} \left( 1 - \frac{\tau}{2} \right) \rho_D c^2, \quad T_{\alpha\alpha} = 0, \quad (20)$$

$$T_{\mu_i\mu_i} \approx \frac{k_N V_{D'} (\gamma_i - 1 + \tau)}{2k_D} \rho_D c^2. \quad (21)$$

The equation for the 00-component shows that the parameter  $\tau$  cannot be equal to 2 because for  $\tau = 2$  we get  $T_{00} = 0$ , what corresponds to the uninteresting case of absence of matter. Moreover,  $T_{00}^0 = \varepsilon$  is the energy density of matter. Therefore, up to the terms of the order of  $1/c^2$ , we have  $T_{00} \approx \varepsilon \approx \rho_D c^2$ . This requires the following relation between Newtonian and multidimensional gravitational constants [16]:

$$k_N = \frac{2}{2 - \tau} \kappa_D / V_{D'} \implies 4\pi G_N = \frac{2}{2 - \tau} S_D \tilde{G}_D / V_{D'}. \quad (22)$$

In the particular case of a point-like massive source, this relation was given in [1, 15]. From Eqs. (20) and (21) we also obtain the relation

$$T_{\mu_i\mu_i} \approx \frac{\gamma_i - 1 + \tau}{2 - \tau} T_{00}. \quad (23)$$

Taking into account that, up to the terms of the order of  $1/c^2$ , components  $T_{\mu_i\mu_i}$  define pressure in the  $i$ -th internal space ( $T_{\mu_i\mu_i} \approx p_i$ ), we get from Eq. (23) the following

equations of state in these spaces:

$$p_i = \frac{\gamma_i - 1 + \tau}{2 - \tau} \varepsilon, \quad i = 1, \dots, N. \quad (24)$$

Since  $T_{11} = T_{22} = T_{33} = 0$ , in our three-dimensional space we have dust-like equation of state, namely:  $p_0 = 0$ . In the case of a point-like mass, the parameters  $\gamma_i$  satisfy the condition (8). It can be easily seen that for these values of  $\gamma_i$ , all  $T_{\mu_i \mu_i}$  are equal to zero. Therefore, in this case,  $T_{00}$  is the only non-zero component in the external space, as well as in all internal spaces, and we have the same dust-like equations of state in all spaces, namely:  $p_i = 0$ ,  $i = 0, \dots, N$ .

#### IV. EXPERIMENTAL RESTRICTIONS ON SOLITONS. LATENT SOLITONS

In this section, we want to get the experimental restrictions for the parameters  $\gamma_i$ . This can be done with the help of the parameterized post-Newtonian (PPN) formalism. According to the PPN formalism (see, e.g., [17, 18]), the four-dimensional static spherically symmetric line element in isotropic coordinates is parameterized as follows:

$$ds^2 = \left(1 - \frac{r_g}{r_3} + \beta \frac{r_g^2}{2r_3^2}\right) c^2 dt^2 - \left(1 + \gamma \frac{r_g}{r_3}\right) \sum_{i=1}^3 (dx^i)^2. \quad (25)$$

In general relativity we have  $\beta = \gamma = 1$ . To get  $\beta$  and  $\gamma$  in the case of the soliton solution (1), it is sufficient to compare the metric coefficients in Eq. (25) with the corresponding asymptotic expressions (10) and (11), what immediately gives the soliton PPN parameters

$$\beta_s = 1, \quad \gamma_s = 1 - \tau. \quad (26)$$

With the help of these PPN parameters, we can easily get formulas for the famous gravitational experiments [5, 17, 19]:

(i) *Perihelion shift*

$$\begin{aligned} \delta\psi &= \frac{6\pi m G_N}{\lambda(1-e^2)} \frac{1}{c^2} \frac{1}{3} (2 + 2\gamma_s - \beta_s) \\ &= \frac{6\pi m G_N}{\lambda(1-e^2)} \frac{3-2\tau}{3} = \frac{\pi r_g}{\lambda(1-e^2)} (3-2\tau), \end{aligned} \quad (27)$$

where  $\lambda$  is the semi-major axis of the ellipse and  $e$  is its eccentricity.

(ii) *Deflection of light*

$$\delta\psi = (1 + \gamma_s) \frac{r_g}{\rho} = (2 - \tau) \frac{r_g}{\rho}, \quad (28)$$

where  $\rho$  is the distance of closest approach (impact parameter) of the rays path to the gravitating mass  $m$ .

(iii) *Time delay of radar echoes (Shapiro time-delay effect)*

$$\begin{aligned} \delta t &= (1 + \gamma_s) \frac{r_g}{c} \ln \left( \frac{4r_{Earth} r_{planet}}{R_{Sun}^2} \right) \\ &= (2 - \tau) \frac{r_g}{c} \ln \left( \frac{4r_{Earth} r_{planet}}{R_{Sun}^2} \right). \end{aligned} \quad (29)$$

Comparison of the formulas (27)-(29) with experimental data gives the possibility to restrict parameters of the soliton solutions. In fact, we can also get it directly from experimental restriction on the PPN parameter  $\gamma$ . The tightest constraint on  $\gamma$  comes from the Shapiro time-delay experiment using the Cassini spacecraft, namely:  $\gamma - 1 = (2.1 \pm 2.3) \times 10^{-5}$  [19-21]. Thus, from Eq. (26) we find that the solitonic parameter  $\tau$  should satisfy the condition [22]

$$\tau = -(2.1 \pm 2.3) \times 10^{-5}. \quad (30)$$

In the case of the point-like massive soliton described by Eqs. (7)-(9), we have  $\tau = D'/(1 + D') \sim \mathcal{O}(1)$ , what obviously contradicts Eq. (30) (in accordance with the results of [1, 5]).

Equation (26) shows that there is a very interesting class of solitons which are defined by the condition

$$\tau = \sum_{i=1}^N d_i \gamma_i = 0. \quad (31)$$

Counting only with the gravitational experiments mentioned above, it is impossible to differ these Kaluza-Klein solitons from general relativity because they have  $\gamma_s = 1$  as in general relativity [23]. For this reason, we call these solutions *latent solitons*. For these latent solitons, equations of state (24) in the internal spaces are reduced to

$$p_i = \frac{\gamma_i - 1}{2} \varepsilon, \quad i = 1, \dots, N. \quad (32)$$

Black strings ( $N = 1$ ,  $d_1 = 1$ ) and black branes ( $N > 1$ ) are characterized by the condition that all  $\gamma_i = 0$ ,  $i \geq 1$ . Obviously, they belong to the class of latent solitons and they have the equations of state

$$p_i = -\frac{1}{2} \varepsilon, \quad i = 1, \dots, N. \quad (33)$$

It is known (see, e.g., [1, 6]) that in the case of three-dimensional external space such equations of state are the only ones which do not spoil the condition of the internal space stabilization for the compact astrophysical objects with the dust-like equation of state  $p_0 = 0$  (in the external space). Therefore, non-zero parameters  $\gamma_i$  can be treated as a measure of the latent soliton destabilization [24].

We would like to stress the following: It is well known that black strings/branes have the topology (4-dimensional Schwarzschild spacetime)  $\times$  (flat internal

spaces). In this case, it does not seem surprising that gravitational experiments lead to the same results as for general relativity. However, the latent solitons, in the general case, do not have either Schwarzschildian metrics for 4-dimensional part of spacetime, nor flat metrics for the extra dimensions. Nevertheless, within the considered accuracy, it is also impossible to distinguish them from general relativity. This is really surprising.

To conclude this section, we would like to mention that the relation between Newtonian and multidimensional gravitational constants for latent solitons is reduced to the following equation:

$$4\pi G_N = S_D \tilde{G}_D / V_{D'}. \quad (34)$$

## V. EXPERIMENTAL RESTRICTIONS ON THE EQUATIONS OF STATE OF A MULTIDIMENSIONAL PERFECT FLUID

In this section, we want to show that for static spherically symmetric perfect fluid with dust-like equation of state in the external space, the condition  $h_{00} = h_{\alpha\alpha}$  (which provides the agreement with the gravitational experiments at the same level of accuracy as general relativity) results in the latent soliton condition (31), and equations of state (32), together with the condition  $R_{\mu_i\mu_i} = 0 \implies h_{\mu_i\mu_i} = 0$ , leads to the stability condition (33) and singles out  $d_0 = 3$  for the number of the external dimensions.

Let us consider a static spherically symmetric perfect fluid with energy-momentum tensor

$$T^i_k = \text{diag} \left( \varepsilon, \underbrace{-p_0, \dots, -p_0}_{d_0 \text{ times}}, \underbrace{-p_1, \dots, -p_1}_{d_1 \text{ times}}, \dots, \underbrace{-p_N, \dots, -p_N}_{d_N \text{ times}} \right). \quad (35)$$

We recall that we are using the notations:  $i, k = 0, 1, \dots, D$ ;  $a, b = 1, \dots, D$ ;  $\alpha, \beta = 1, \dots, d_0$  and  $\mu_i = 1 + \sum_{j=0}^{i-1} d_j, \dots, d_i + \sum_{j=0}^{i-1} d_j$ ,  $i = 1, \dots, N$ . For static spherically symmetric configurations we have  $g_{0a} = 0$  and  $g_{ab} = 0$ ,  $a \neq b$ . Since we want to apply this model to ordinary astrophysical objects, where the condition  $T^0_0 \gg |T^\alpha_\alpha|$  usually holds, we assume the dust-like equation of state in the  $d_0$ -dimensional external space, namely  $p_0 = 0$ , but we leave equations of state arbitrary in the  $i$ -th internal space, namely  $p_i = \omega_i \varepsilon$ . Obviously,  $\varepsilon$  is equal to zero outside the compact astrophysical objects. Moreover, we consider the weak-field approximation, in which the metric coefficients can be expressed in the form

$$g_{00} \approx 1 + h_{00}, \quad g_{aa} \approx -1 + h_{aa}, \quad h_{00}, h_{aa} \sim O(1/c^2). \quad (36)$$

As an additional requirement, we impose that the considered configuration does not contradict the observations. It will be so if the following conditions hold:  $h_{00} = h_{\alpha\alpha}$  and  $h_{\mu_i\mu_i} = 0$  (see Ref. [1]). In what follows, we define

which equations of state are obtained as a result of these restrictions.

Taking into account that  $T = \sum_{i=0}^D T^i_i = \varepsilon(1 - \sum_{i=1}^N \omega_i d_i)$ ,  $T_{\alpha\alpha} = 0$ ,  $\varepsilon \sim O(c^2)$ , and, up to terms of the order of  $c^2$ , that  $T_{00} \approx T^0_0$ ,  $T_{\mu_i\mu_i} \approx -T^{\mu_i}_{\mu_i}$ , we get from the Einstein equation (16) the non-zero components of Ricci tensor (up to the order of  $1/c^2$ ):

$$R_{00} \approx \frac{\varepsilon k_D}{D-1} \left[ d_0 - 2 + \sum_{i=1}^N d_i(1 + \omega_i) \right], \quad (37)$$

$$R_{\alpha\alpha} \approx \frac{\varepsilon k_D}{D-1} \left[ 1 - \sum_{i=1}^N d_i \omega_i \right], \quad (38)$$

$$R_{\mu_i\mu_i} \approx \frac{\varepsilon k_D}{D-1} \times \left[ \omega_i \left( \sum_{j=0}^N {}'d_j - 1 \right) + 1 - \sum_{j=1}^N {}'d_j \omega_j \right], \quad (39)$$

where  $k_D \sim O(1/c^4)$  is defined in Sec. III, and the prime in the summation of Eq. (39) means that we must not take into account the  $i$ -th term. Equations (37) and (38) show that the  $R_{00}$  and  $R_{\alpha\alpha}$  components are related as follows:

$$R_{\alpha\alpha} = \frac{1 - \sum_{i=1}^N d_i \omega_i}{d_0 - 2 + \sum_{i=1}^N d_i(1 + \omega_i)} R_{00}. \quad (40)$$

On the other hand, in the weak-field limit the components of Ricci tensor read

$$R_{00} \approx \frac{1}{2} \Delta h_{00}, \quad R_{aa} \approx \frac{1}{2} \Delta h_{aa}, \quad a = 1, \dots, D, \quad (41)$$

where as usual we can put  $h_{00} \equiv 2\varphi/c^2$ , and  $\Delta$  is  $D$ -dimensional Laplace operator defined in Eqs. (13)-(15). Therefore, from Eqs. (40) and (41) we obtain

$$h_{\alpha\alpha} = \frac{1 - \sum_{i=1}^N d_i \omega_i}{d_0 - 2 + \sum_{i=1}^N d_i(1 + \omega_i)} h_{00}, \quad \alpha = 1, \dots, d_0. \quad (42)$$

As we have mentioned above, to be in agreement with experiments we should demand  $h_{\alpha\alpha} = h_{00}$ , what leads to the following restriction on the parameters  $\omega_i$  of the equations of state:

$$3 - d_0 - \sum_{i=1}^N d_i = 2 \sum_{i=1}^N d_i \omega_i. \quad (43)$$

In the case of three-dimensional external space ( $d_0 = 3$ ), this constraint is reduced to

$$\sum_{i=1}^N d_i \left( \omega_i + \frac{1}{2} \right) = 0. \quad (44)$$

If we parameterize

$$\omega_i = \frac{\gamma_i - 1}{2}, \quad i = 1, \dots, N, \quad (45)$$

then we arrive at the latent soliton condition (31). Therefore, *the demand that multidimensional perfect fluid with dust-like equation of state ( $p_0 = 0$ ) in the external space provides the same results for gravitational experiments as general relativity, leads to the latent soliton equations of state (32) in the internal spaces.* However, it is known (see Refs. [1, 6]) that the internal spaces can be stabilized if multidimensional perfect fluid with  $p_0 = 0$  has the same equations of state  $\omega_i = -1/2$  in all internal spaces and the external space is three-dimensional ( $d_0 = 3$ ). In other words, it takes place if all  $\gamma_i = 0$  in Eq. (45). Let us show that the additional requirement  $R_{\mu_i\mu_i} = 0$  ensures the fulfillment of these conditions. Indeed, from Eq. (39) we get

$$R_{\mu_i\mu_i} = 0 \implies \omega_i = -\frac{1}{2}, \quad i = 1, \dots, N, \quad (46)$$

where we used the constraint (43) [25]. Now, the substitution  $\omega_i = -1/2$  in Eq. (43) singles out  $d_0 = 3$ . Therefore, *the demand of the internal space stabilization leads, for multidimensional perfect fluid with  $p_0 = 0$ , to the black string/brane equations of state (33) in the internal spaces and, additionally, it selects uniquely the number of the external spaces to be  $d_0 = 3$ .*

To conclude the consideration of this perfect fluid, we want to get the metric coefficients up to  $\mathcal{O}(1/c^2)$  [see Eq. (36)]. To do so, it is sufficient to define the function  $\varphi \equiv h_{00}c^2/2$ . It can be easily seen from Eqs. (37) and (41) that this function satisfies the equation

$$\Delta\varphi = \frac{c^2}{2}\Delta h_{00} \approx c^2 R_{00} \approx S_D \tilde{G}_D \rho_D, \quad (47)$$

where we have used the constraint (43) for arbitrary  $d_0$  and the relation  $\varepsilon \approx \rho_D c^2$ . Therefore, to get the metric coefficients we need to solve this equation with proper boundary conditions. We want to reduce this equation to ordinary Poisson equation in three-dimensional external space ( $d_0 = 3$ ). To do so, we consider the case in which matter is uniformly smeared over the extra dimensions, then  $\rho_D = \rho_3/V_{D'}$  (see Sec. III). In this case the non-relativistic potential  $\varphi$  depends only on our external coordinates and  $\Delta$  is reduced to three-dimensional Laplace operator  $\Delta_3$ . Therefore, Eq. (47) is reduced to

$$\Delta_3\varphi \approx (S_D \tilde{G}_D/V_{D'})\rho_3 = 4\pi G_N \rho_3, \quad (48)$$

where we have used the relation (34) between Newtonian and multidimensional gravitational constants. Equation (48) is the usual Poisson equation. It is worth noting that  $\rho_3 = 0$  outside the compact astrophysical object and it is necessary to solve Eq. (48) inside and outside of the object, and to match these solutions at the boundary.

## VI. CONCLUSIONS

In the first part of our investigations (Sec. II-IV), we considered the most general (known to us) soliton solution in Kaluza-Klein models with toroidal compactification of the extra dimensions. Here, each  $d_i$ -dimensional

torus has its own scale factor  $C_i(r_3)$ ,  $i = 1, \dots, N$ , which is characterized by the parameter  $\gamma_i$ . A distinctive feature of these solutions is that their metric coefficients depend only on the length of a radius vector  $r_3$  of the external (our) space [26]. This happens when the matter source is uniformly smeared over the extra dimensions. In this case, the non-relativistic gravitational potential exactly coincides with the Newtonian one. Among the soliton solutions, we sorted out one which corresponds to a point-like mass. This solution is of special interest because it generalizes the well known point-like mass approach of general relativity, which works very well to describe the known gravitational experiments. Then, we investigated the weak-field limit and obtained (in the general case) the equations of state for the soliton matter source. These equations show that in the case of a point-like mass,  $T_{00}$  is the only non-zero component of the energy-momentum tensor, and the equations of state in the external and internal spaces correspond to dust ( $p_i = 0$ ,  $i = 0, \dots, N$ ). We also used the weak-field limit to get the experimental restrictions on the parameters of the soliton solutions. To get it, we found the parameterized post-Newtonian (PPN) parameters  $\beta_s$  and  $\gamma_s$  for the soliton solutions. This gave us a possibility to derive formulas for perihelion shift, deflection of light and time delay of radar echoes. For soliton solutions, the parameter  $\beta_s = 1$  coincides with the one in general relativity. However, the parameter  $\gamma_s = 1 - \tau = 1 - \sum_{i=1}^N d_i \gamma_i$  is different from its general relativistic value ( $\gamma = 1$ ). The PPN parameter  $\gamma$  is strongly restricted by the Shapiro time-delay measurements obtained from the Cassini spacecraft. With the help of this bound, we obtained the limitation on the soliton parameter, namely:  $\tau = -(2.1 \pm 2.3) \times 10^{-5}$ . The point-like mass soliton considerably contradicts this restriction. Obviously, solutions with  $\tau = \sum_{i=1}^N d_i \gamma_i = 0$  satisfy this bound. This is a new class of soliton solutions. For them we have  $\gamma_s = 1$  as in general relativity, and therefore it is impossible to differ experimentally these Kaluza-Klein solitons from general relativity. For this reason we call these solutions *latent solitons*. For the latent solitons, the non-relativistic equations of state are  $p_0 = 0$  in the external (our) space and  $p_i = [(\gamma_i - 1)/2]\varepsilon$  in the internal spaces. All these results were obtained for the realistic case of three-dimensional external space. We would like to stress once again that latent solitons satisfy the gravitational experiments mentioned above at the same level of accuracy as general relativity. Black strings ( $N = 1$ ,  $d_1 = 1$ ) and black branes ( $N > 1$ ) are characterized by the condition that all  $\gamma_i = 0$ ,  $i \geq 1$ . Obviously, they belong to the class of latent solitons and they have the equations of state  $p_i = -\varepsilon/2$ ,  $i = 1, \dots, N$ . It is known (see Refs. [1, 6]) that in the case of three-dimensional external space such equations of state are the only ones which do not spoil the condition of the internal space stabilization for the compact astrophysical objects with the dust-like equation of state ( $p_0 = 0$ ) in the external space.

In the second part of our investigations (Sec. V), we considered a multidimensional static spherically symmetric perfect fluid with dust-like equation of state ( $p_0 = 0$ ) in the external space and arbitrary equations of state ( $p_i = \omega_i \varepsilon$ ) in the internal spaces. The number of external spatial dimensions  $d_0$  was left arbitrary. In the case  $d_0 = 3$ , such perfect fluid describes observable astrophysical objects (e.g., the Sun) in Kaluza-Klein models. We performed our analysis in the weak-field limit where the metric coefficients can be expressed in the form  $g_{00} \approx 1 + h_{00}$ ,  $g_{aa} \approx -1 + h_{aa}$ ,  $a = 1, \dots, D$ , and  $h_{00}, h_{aa} \sim O(1/c^2)$ . (We recall that  $h_{\alpha\alpha}$ ,  $\alpha = 1, \dots, d_0$ , describe perturbations in the external space and  $h_{\mu_i\mu_i}$ ,  $i = 1, \dots, N$ , describe perturbations in the internal spaces). We have shown that the demand of agreement with the gravitational experiments at the same level of accuracy as general relativity,  $h_{00} = h_{\alpha\alpha}$ , results in a constraint for the parameters  $\omega_i$ , which exactly coincides with the latent soliton condition,  $\sum_{i=1}^N d_i \gamma_i = 0$ , in the case  $d_0 = 3$ . In other words, for  $d_0 = 3$ , the equations of state in the internal spaces are  $p_i = [(\gamma_i - 1)/2]\varepsilon$ . The additional requirement  $h_{\mu_i\mu_i} = 0$ , together with the previous condition  $h_{00} = h_{\alpha\alpha}$ , (i) leads to the equation  $\omega_i = -1/2$  (i.e.  $\gamma_i = 0$ ) and (ii) singles out  $d_0 = 3$  [27]. Therefore, we arrived at the black string/brane equations of state in the internal spaces. Precisely these equations of state (supplemented by the condition  $p_0 = 0$ ) satisfy the necessary condition of the internal space stabilization in the case  $d_0 = 3$  [1, 6]. We see that the first and second parts of our investigations agree with each other, as expected.

We can summarize the main conclusion of our paper as follows. For compact astrophysical objects with dust-like equation of state in the external space ( $p_0 = 0$ ),

the demand of the agreement with the gravitational experiments requires the condition (30), namely:  $\tau = -(2.1 \pm 2.3) \times 10^{-5}$ . However, to be at the same level of accuracy as general relativity, we must have  $\tau = 0$ . In other words, we should consider the latent solitons with equations of state (32) in the internal spaces (in the case  $d_0 = 3$ ). Moreover, the condition of stability of the internal spaces singles out black strings/branes from the latent solitons and leads uniquely to  $p_i = -\varepsilon/2$  as the black string/brane equations of state in the internal spaces, and to the number of the external dimensions  $d_0 = 3$ . The main problem with the black strings/branes is to find a physically reasonable mechanism which can explain how the ordinary particles forming the astrophysical objects can acquire rather specific equations of state ( $p_i = -\varepsilon/2$ ) in the internal spaces.

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  - [11] In the literature, a different parameterization (in terms of parameters  $k$  and  $\varepsilon$ ) is commonly used (see, e. g., Ref. [4]) in the case of one internal space  $N = 1$ . The relation between both of these parameterizations is the following:  $\theta = \varepsilon k$  and  $\gamma_1 \equiv \gamma = 1/k$ . Then,  $\theta\gamma = \varepsilon$  and  $\theta(1 - \tau) = \varepsilon(k - d)$ , where  $d \equiv d_1$ .
  - [12] A particular soliton solution was also found in Ref. [13]. In the case of four external spacetime coordinates and  $D'$  internal spatial coordinates, the parameters in the corresponding solution (25) in Ref. [13] are:  $q = 2$ ,  $p = D'$  and  $\Delta^2 = 2(D' + 1)/(D' + 2)$ . If we rewrite this solution in the isotropic coordinates, then we immediately get the connection with our parameters, namely:  $\gamma_i = -1$ ,  $i = 1, \dots, N$ ,  $\tau = -D'$ ,  $\sigma = D'$  and  $\theta = \sqrt{2/[(D' + 1)(D' + 2)]}$ .
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- [16] It is worth noting the singular nature of  $\tau = 2$ , which follows from Eq. (20), as we require the  $T_{00}$  component of the energy-momentum tensor to coincide with the energy density  $\rho_D c^2$  in the non-relativistic limit. Such relationship between  $T_{00}$  and  $\rho_D c^2$  provides the correct transition to the Newtonian approximation. This physical condition leads to equation (22), which diverges for  $\tau = 2$ .
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- [22] In the case of one  $d$ -dimensional internal space ( $N = 1$ ), we can rewrite this bound with respect to the parameter  $k$  defined in [11] as  $|k| \geq d \times 2.3 \times 10^4$ . This inequality shows that the increase of the number of internal space dimensions  $d$  imposes stronger restrictions on  $k$ .
- [23] It can be easily seen from Eqs. (10)-(12) that the parameter  $\tau$  defines also the difference between perturbations  $h_{00}$  and  $h_{\alpha\alpha}$ :  $h_{00} - h_{\alpha\alpha} = -\tau r_g / r_3$ . Precisely because of this difference, gravitational experiments in Kaluza-Klein models and in general relativity lead to different results. When  $\tau \rightarrow 0$ , this difference disappears. The additional limit  $\gamma_i \rightarrow 0 \Rightarrow h_{\mu_i \mu_i} \rightarrow 0$  provides stabilization of the internal spaces [1]. It is worth noting that this instability is different from the Gregory-Laflamme one for black strings, as it clearly follows from appendix in Ref. [1].
- [24] If we rewrite equations of state in the form  $p_i = (\alpha_i - 1)\varepsilon$ ,  $i = 0, \dots, N$ , then for the latent solitons we have  $\alpha_0 = 1$ ,  $\alpha_i = (1 + \gamma_i)/2$ ,  $i = 1, \dots, N$ . For these values of  $\alpha_0$  and  $\alpha_i$ , we get on the right-hand side of Eq. (A15) in Ref. [1] the terms  $(\gamma_i d_i / 2) \kappa_N \rho_3$ . These terms are dynamical functions because of the dynamical behavior of the energy density  $\rho_3$ . This results in a violation of the necessary condition for the internal space stabilization.
- [25] It can be also easily seen that  $R_{\mu_i \mu_i} = 0 \Rightarrow \Delta h_{\mu_i \mu_i} = 0$ , what, together with the boundary conditions (finiteness of  $h_{\mu_i \mu_i}$  at  $r_3 = 0$  and  $h_{\mu_i \mu_i} \rightarrow 0$  for  $r_3 \rightarrow +\infty$ ), gives  $h_{\mu_i \mu_i} = 0$ .
- [26] We assumed the external space to be three-dimensional ( $d_0 = 3$ ), because we want to establish the relation between the considered solitons and the observations.
- [27] It is worth noting that in Ref. [28] there is also a suggestion of explanation why our world is three-dimensional. The reasoning is based on the behavior of isotropic multidimensional Friedmann universes filled with a gas of branes. The authors have arrived at the conclusion that such universe will naturally come to be dominated by 3-branes and 7-branes. Obviously, our conclusion about the three-dimensionality of our Universe is based on a completely different (anisotropic) multidimensional model.
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